1) $\left(\begin{array}{cccc|c} 3 & 2 & 3 & -2 & 1 \\ 1 & 1 & 1 & 0 & \alpha \\ 1 & 2 & 1 & -1 & 2 \end{array}\right)$

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$\rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 1 & 1 & 1 & 0 & \alpha \\ 3 & 2 & 3 & -2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & -1 & 0 & 1 & \alpha-2 \\ 0 & -4 & 0 & 1 & -5 \end{array}\right)$

$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2\alpha-2 \\ 0 & -1 & 0 & 1 & \alpha-2 \\ 0 & 0 & 0 & -3 & -4\alpha+3 \end{array}\right)$

$\Rightarrow x_4 = \frac{4}{3}\alpha - 1$, $\quad x_2 = x_4 + 2 - \alpha$
$= \frac{1}{3}\alpha + 1$

$x_1 = -x_3 - x_4 + 2\alpha - 2 = -x_3 - \frac{4}{3}\alpha + 1 + 2\alpha - 2$

$= -x_3 + \frac{2}{3}\alpha - 1$

$\Rightarrow x = \begin{pmatrix} -x_3 + \frac{2}{3}\alpha - 1 \\ \frac{1}{3}\alpha + 1 \\ x_3 \\ \frac{4}{3}\alpha - 1 \end{pmatrix}$

Consistent for all $\alpha \in \mathbb{R}$.

$= x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{2}{3}\alpha - 1 \\ \frac{1}{3}\alpha + 1 \\ 0 \\ \frac{4}{3}\alpha - 1 \end{pmatrix}$

$\boxed{2}$

2 a) $\det\begin{pmatrix} 4 & 3 & 1 & 2 \\ 1 & 9 & 0 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{pmatrix} = -\det\begin{pmatrix} 1 & 9 & 0 & 2 \\ 4 & 3 & 1 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{pmatrix}$

$= -\det\begin{pmatrix} 1 & 9 & 0 & 2 \\ 0 & -33 & 1 & -6 \\ 0 & -69 & 2 & -18 \\ 0 & -33 & 1 & -7 \end{pmatrix}$

$= -\det\begin{pmatrix} -33 & 1 & -6 \\ -69 & 2 & -18 \\ -33 & 1 & -7 \end{pmatrix}$

$= -\det\begin{pmatrix} -33 & 1 & -6 \\ -69 & 2 & -18 \\ 0 & 0 & -1 \end{pmatrix}$

$= \det\begin{pmatrix} -33 & 1 \\ -69 & 2 \end{pmatrix} = -66 + 69 = 3$

2b) $\det(-2A) = (-2)^4 \det A = 16 \cdot 3 = 48$

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3). $\det(AB) = \det(-BA)$ since $AB = -BA$. 3

But $\det(-BA) = (-1)^n \det(BA)$

$= -(\det B)(\det A)$

Since n is odd. It follows that

$(\det A)(\det B) = \det(AB) = -\det(BA) = -(\det B)(\det A)$

$\Rightarrow (\det A)(\det B) = 0$

$\Rightarrow \det A = 0$ or $\det B = 0$

$\Rightarrow A$ or B is not invertible.

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4) $p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$

$= (1-\lambda)^3$

$\Rightarrow \lambda = 1$ is the only eigenvalue. Now find the eigenvectors by solving $(A - 1I)x = 0$.

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow x_2 = -x_3 \Rightarrow x = \begin{pmatrix} x_1 \\ -x_3 \\ x_3 \end{pmatrix}$$

$$\Rightarrow x = x_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

So $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$ is a basis for the eigenspace

Since algebraic multiplicity $= 3$ and geometric multiplicity $= 2$

A is not ~~[illegible]~~ diagonalizable.

5) $x' = Ax$, $x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $\boxed{5}$

$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$. Diagonalize A.

$p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix}$

$= -(3-\lambda)(2+\lambda) + 4 = -6 - \lambda + \lambda^2 + 4$

$= (\lambda - 2)(\lambda + 1)$

$\underline{\lambda_1 = 2}$ Solve $(A - 2I)v_1 = 0$.

$\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 = 2x_2$

$\Rightarrow v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$\underline{\lambda_2 = -1}$ Solve $(A + I)v_2 = 0$

$\left(\begin{array}{cc|c} 4 & -2 & 0 \\ 2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_2 = 2x_1$

$\Rightarrow v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$A = QDQ^{-1}$ where $D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$Q^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

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5 continued) $x(t) = e^{At}x(0) = Qe^{Dt}Q^{-1}x(0)$ 6

$e^{Dt} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}$, $Q^{-1}x(0) = \frac{1}{3}\begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow x(t) = e^{2t}v_1 - e^{-t}v_2 = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

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6) Use induction to show that $A^k \in \operatorname{span}\{I, A, \dots, A^{n-1}\}$.

This is true when $k=n$ by Cayley-Hamilton:

$(-1)^n A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 I = 0$ where

$p(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \dots + a_1 t + a_0$ is the characteristic polynomial.

Now assume it is true for $k > n$ and prove it for $k+1$. We have

$$A^{k+1} = A A^k = A[\underbrace{b_{n-1}A^{n-1} + \dots + b_1 A + b_0 I}_{\text{by induction hypothesis}}]$$

$$= b_{n-1}A^n + \dots + b_1 A^2 + b_0 A$$

$$= b_{n-1}(-1)^{1+n}[a_{n-1}A^{n-1} + \dots + a_1 A + a_0 I]$$
$$+ b_{n-2}A^{n-1} + \dots + b_1 A^2 + b_0 A$$

$$\in \operatorname{Span}\{I, \dots, A^{n-1}\}.$$

7) We have already found the eigenvalues $\lambda = 1$ and eigenvectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

Need to find one more generalized eigenvector.

Solve $(A - I)v_3 = \alpha v_1 + \beta v_2$ for some α, β.

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & \alpha \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & 0 & \beta \end{array}\right) \Rightarrow \beta = 0, \quad \alpha = 1$$

$$x_2 = -x_3 + 1.$$

$$\Rightarrow v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad Av_3 = v_3 + v_1.$$

$$, \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$